

NEW EXAMPLES OF DETERMINANT DIVISIBILITY SEQUENCES

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ABSTRACT. In this paper we consider divisibility sequences obtained from square matrices. We work with of matrix divisibility sequences associated to a semigroup and arising from endomorphisms of an affine space. We prove that determinant divisibility sequences originated from powers of square matrices are generalized Lucas sequences.

1. INTRODUCTION

By the divisibility sequence we mean in this paper a sequence $\{d_n\}_{n \in \mathbb{N}}$ of integers such that if $n|m$ then $d_n|d_m$. One of the most famous divisibility sequence is the Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34,... which arise from linear recurrence: $F_n = F_{n-1} + F_{n-2}$. This is an example of the Lucas sequences: $L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where α, β are the roots of some quadratic polynomial over \mathbb{Z} . See [2] for a complete classification of linear recurrence divisibility sequences and [5], [6] for introduction to other divisibility sequences. In this paper we discuss properties of certain matrix divisibility sequences. We follow the approach initiated in [1].

2. MATRIX DIVISIBILITY SEQUENCE

Let S be a commutative ring with 1. Let $M_r(S)$ be a ring of $r \times r$ matrices with entries in S . By a divisor class of a matrix $M \in M_r(S)$ we mean a coset $GL_r(S) \cdot M$ of M with respect to the natural left action of $GL_r(S)$. We say that matrix $M \in M_r(S)$ divides a matrix $N \in M_r(S)$ if there exists a $Q \in M_r(S)$ such that $N = QM$. If M divides N , then any element of the divisor class of M also divides N . Let (Γ, \cdot) denote a semigroup. A *divisibility sequence of matrices* over a commutative ring S , indexed by Γ , is a collection of matrices $\{M_\alpha\}_{\alpha \in \Gamma}$ in $M_r(S)$, such that if α divides β in Γ , then M_α divides M_β in $M_r(S)$. If $\{M_\alpha\}_{\alpha \in \Gamma}$ is a divisibility sequence of matrices, then by the multiplicativity of the determinant $\{\det(M_\alpha)\}_{\alpha \in \Gamma}$ is a divisibility sequence of elements of the ring S .

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We fix a faithful representation:

$$[\cdot] : \Gamma \hookrightarrow \text{End}(\mathbb{A}_S^m) : \alpha \rightarrow [\alpha]$$

of Γ into the group of endomorphisms of affine m -dimensional space \mathbb{A}_S^m over S .

Definition 2.1. Let $x \in \mathbb{A}_S^r$. The *matrix divisibility sequence* associated to $(\Gamma, [\cdot])$ is the sequence of Jacobians $\{J_\alpha(x)\}_{\alpha \in \Gamma}$ which are $r \times r$ matrices with (i, j) -entry given by partial differentials:

$$[J_\alpha(x)]_{i,j} := \partial([\alpha](x))_i / \partial x_j,$$

where $([\alpha](x))_i$ is an i th entry of the value of the endomorphism $[\alpha]$ on x . The associated *determinant divisibility sequence* is defined by $\{\det(J_\alpha(x))\}_{\alpha \in \Gamma}$.

3. MAIN RESULT

Theorem 3.1. Let $X \in GL_r(\mathbb{Z})$ and $\lambda_1, \dots, \lambda_r$ be eigenvalues of X . Then for every $n \geq 1$:

$$(3.1) \quad D_n = n^2 [\det X]^{n-1} \prod_{1 \leq i < j \leq r} \left(\frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j} \right)^2$$

is an integer and the sequence $\{D_n\}_{n \in \mathbb{N}}$ is a determinant divisibility sequence.

Proof: Let X, Y, Z be square $s \times s$ matrices. Assume that entries of matrices Y and Z are functions of entries of the matrix X . Then the following matrix derivative formula holds ([4]):

$$(3.2) \quad \frac{d(YZ)}{dX} = (I \otimes Y) \frac{dZ}{dX} + (Z^t \otimes I) \frac{dY}{dX},$$

where \otimes means the Kronecker product, I is the identity matrix of rank s and A^t means the transpose matrix of A . In addition we will use property of the Kronecker product:

$$(3.3) \quad (A \otimes C)(B \otimes D) = AB \otimes CD$$

for any square matrices A, B, C, D of size $s \times s$. From now on we fix $\Gamma = \mathbb{N}$. Consider the group G of all invertible $s \times s$ matrices with the embedding:

$$G \rightarrow \mathbb{A}^{s^2} : \begin{bmatrix} X_{11} & \cdots & X_{1s} \\ \vdots & \ddots & \vdots \\ X_{s1} & \cdots & X_{ss} \end{bmatrix} \mapsto (X_{11}, \dots, X_{1s}, \dots, X_{s1}, \dots, X_{ss})$$

We define the endomorphism $[n]$ for $n \in \mathbb{N}$. Let $X := [X_{ij}] \in G$ and respectively $X^n := [\bar{X}_{kl}] \in G$, where we treat \bar{X}_{kl} as functions of X_{ij} , for $1 \leq i, j, k, l \leq s$. We define $[n] : \mathbb{A}^{s^2} \rightarrow \mathbb{A}^{s^2}$ as

$$[n](X_{11}, \dots, X_{1s}, \dots, X_{s1}, \dots, X_{ss}) = (\bar{X}_{11}, \dots, \bar{X}_{1s}, \dots, \bar{X}_{s1}, \dots, \bar{X}_{ss}).$$

Using (3.2) we compute Jacobians of the n -th power of the matrix X

$$J_n = \frac{d(X^n)}{dX} = \frac{d(X^{n-1}X)}{dX} = (I \otimes X^{n-1}) \frac{dX}{dX} + (X^t \otimes I) \frac{dX^{n-1}}{dX}.$$

By induction and the property (3.3) of the Kronecker product we get:

$$J_n = \sum_{k=0}^{n-1} (X^t)^k \otimes X^{n-1-k}.$$

Assume that $X \in GL_s(\mathbb{Z})$ is a diagonalizable matrix. Then $X = PDP^{-1}$, for some $P \in GL_s(\mathbb{C})$ and a diagonal matrix $D = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_s\}$. The set of matrices with distinct eigenvalues is dense in the set of all square matrices $M_s(\mathbb{C})$ with respect to the topology of \mathbb{C}^{s^2} . Hence we can assume that X has different eigenvalues. Therefore

$$\begin{aligned} J_n &= \sum_{k=0}^{n-1} ((PDP^{-1})^t)^k \otimes (PDP^{-1})^{n-1-k} = \sum_{k=0}^{n-1} (P^{-1})^t (D^k)^t (P)^t \otimes PD^{n-1-k} P^{-1} = \\ &= ((P^{-1})^t \otimes P) \left[\sum_{k=0}^{n-1} (D^k \otimes D^{n-1-k}) \right] (P^t \otimes P^{-1}) \end{aligned}$$

and the determinant divisibility sequence is of the form:

$$D_n = \det J_n = \det \left[\sum_{k=0}^{n-1} (D^k \otimes D^{n-1-k}) \right].$$

Since $D^k \otimes D^{n-1-k}$ is a diagonal matrix whose diagonal consists of terms $\lambda_i^k \lambda_j^{n-1-k}$, we conclude that:

$$D_n = n^2 \prod_{l=0}^s \lambda_l^{n-1} \prod_{1 \leq i \neq j \leq s} \sum_{k=0}^{n-1} \lambda_i^k \lambda_j^{n-1-k} = n^2 [\det X]^{n-1} \prod_{1 \leq i < j \leq s} \left(\frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j} \right)^2.$$

The last term on the right hand side is a product of values of symmetric polynomials computed at eigenvalues of the matrix X . The Galois group of the splitting field of the characteristic polynomial of X acts trivially on these algebraic integers, hence $D_n \in \mathbb{Z}$. For any n, m such that n divides m we have $[\det X]^n | [\det X]^m$ and $(\lambda_i^n - \lambda_j^n) | (\lambda_i^m - \lambda_j^m)$. Therefore, the sequence D_n is a divisibility sequence. For equal eigenvalues we compute D_n using the exact form of symmetric polynomials instead of their fractional expression.

If X is not a diagonalizable matrix, then instead of the matrix D we consider an upper-triangular matrix obtained from the Jordan form of X with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ on the diagonal and eventually we get the same sequence D_n .

4. EXAMPLES

1) Let $X \in \text{GL}_2(\mathbb{Z})$ and $a = \text{tr}X, b = \text{tr}^2X - 4\det X$. Then using Theorem 1 we obtain the sequence presented in [1], example 4.3:

$$D_n = \frac{n^2}{b} [\det X]^{n-1} \left(\left(\frac{a + \sqrt{b}}{2} \right)^n - \left(\frac{a - \sqrt{b}}{2} \right)^n \right)^2$$

2) Let $X \in \text{GL}_3(\mathbb{Z})$ and $b = -\text{tr}X, c = X_{11} + X_{22} + X_{33}, d = -\det X$. The discriminant of the characteristic polynomial of X is $\Delta = (4\Delta_0^3 - \Delta_1^2)/27$, where $\Delta_0 = b^2 - 3c$ and $\Delta_1 = 2b^3 - 9bc + 27d$. We obtain the divisibility sequence defined by:

$$D_n = \frac{n^2 d^{n-1}}{\Delta} \prod_{i=1}^3 \left[\left(\frac{b + \epsilon^i A + \epsilon^{2i} \Delta_0 \bar{A}}{3} \right)^n - \left(\frac{b + \epsilon^{i+1} A + \epsilon^{2(i+2)} \Delta_0 \bar{A}}{3} \right)^n \right]^2,$$

where $A = \sqrt[3]{(\Delta_1 + \sqrt{-27\Delta})/2}, \bar{A} = \sqrt[3]{(\Delta_1 - \sqrt{-27\Delta})/2}$ and ϵ is a fixed primitive cube root of unity.

3) It is easy to compute values of D_n for any square matrix X . The matrix $X = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 1 & 3 \\ -1 & 0 & 1 \end{bmatrix} \in \text{GL}_3(\mathbb{Z})$ gives the following divisibility sequence:

n	d_n	factorization of d_n
1	1	1
2	100	$2^2 5^2$
3	6561	3^8
4	193600	$2^6 5^2 11^2$
5	808201	$29^2 31^2$
6	189612900	$2^2 3^8 5^2 17^2$
7	50131657801	$41^2 43^2 127^2$
8	4096576000000	$2^{12} 5^6 11^2 23^2$
9	159625511221401	$3^{14} 53^2 109^2$
10	1865976489302500	$2^2 5^4 29^2 31^6$
11	31583922467632921	$131^2 857^2 1583^2$
12	21985833099924302400	$2^6 3^8 5^2 11^2 17^2 71^2 109^2$
13	2370466451421685365841	$1637^2 4057^2 7331^2$
14	118070682478980566428900	$2^2 5^2 41^2 43^6 83^2 127^2$
15	2362255369723766871090801	$3^8 29^2 31^2 2969^2 7109^2$
16	84956038709284864000000	$2^{18} 5^6 11^2 23^2 47^2 383^2$

4) The matrix $X = \begin{bmatrix} -1 & 2 & 4 & -1 \\ 0 & 1 & -2 & 2 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in Gl_4(\mathbb{Z})$ gives the following divisibility sequence:

n	d_n	factorization of d_n
1	1	1
2	65536	2^{16}
3	1	1
4	281474976710656	2^{48}
5	18448995933652254721	4295229439^2
6	18013780039499776	$2^{16}7^274897^2$
7	7922332684788105606123945 9841	281466386710529^2
8	5194832314440011219064571 543158784	$2^{60}7^423^259561^2$
9	5775028020578783636854257 0774529	$37^2701^2292993041329^2$
10	2229666183092939997026658 7262959037621272576	$2^{16}19^43449^44295229439^2$
11	1463330673647120201450844 900178197550156472647681	$32363^27282397^25132726390881^2$
12	9132817257932632786870155 6304335790376407269376	$2^{48}7^213^210177^274897^2259691^2$
13	6274228310768040852924579 1977173633010223354344560 089620481	$3^{18}3769^215053^227205307^22607270173^2$
14	4124060734576866740544330 9242707472643430878237415 8659336863744	$2^{16}13^2794009^227304061^2281466386710529^2$
15	9806175554643239147044270 0484791252942607177394577 588251525121	$17489^24295229439^2131825214490835791^2$
16	1765121615339370515604475 3103664126594001046686372 42611924881667927310336	$2^{72}7^823^259561^220394769^2288208447^2$

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